

## BEATING THE MEMORY

[Formulae, Properties and Results to be remembered from all the chapters at a glance]

### Unit - I : FOURIER SERIES

- *Fourier series of period  $2\pi$  and Euler's formulae for the Fourier coefficients  $a_0, a_n, b_n$*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \dots \quad \text{[Fourier series]}$$

$$a_0 = \frac{1}{\pi} \int_c^{c+2\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx dx \quad \text{[Euler's formulae]}$$

- *Fourier series of arbitrary period  $2l$  and the related Euler's formulae*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \text{[Fourier series]}$$

$$a_0 = \frac{1}{l} \int_c^{c+2l} f(x) dx$$

$$a_n = \frac{1}{l} \int_c^{c+2l} f(x) \cos \frac{n\pi x}{l} dx$$

$$b_n = \frac{1}{l} \int_c^{c+2l} f(x) \sin \frac{n\pi x}{l} dx \dots \quad \text{[Euler's formulae]}$$

- *Fourier coefficients in the case of even and odd nature of  $f(x)$ .*

Interval	$f(x)$ is even			$f(x)$ is odd		
	$a_0$	$a_n$	$b_n$	$a_0$	$a_n$	$b_n$
$(-\pi, \pi)$ or $(0, 2\pi)$	$\frac{2}{\pi} \int_0^{\pi} f(x) dx$	$\frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$	0	0	0	$\frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$
$(-l, l)$ or $(0, 2l)$ type of intervals	$2 \int_0^l f(x) dx$	$2 \int_0^l f(x) \cos \frac{n\pi x}{l} dx$	0	0	0	$2 \int_0^l f(x) \sin \frac{n\pi x}{l} dx$

Note :  $f(x)$  is even implies  $b_n = 0$  and  $f(x)$  is odd implies  $a_0 = 0 = a_n$

➤ Half range Fourier series (cosine / sine) along with the related formulae for the Fourier coefficients.

$f(x)$ in	Requirement	Series	Fourier Coefficients
$(0, \pi)$	Cosine series	$\frac{a_0}{2} + \sum_1^{\infty} a_n \cos nx$	$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$ $a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$
$(0, \pi)$	Sine series	$\sum_1^{\infty} b_n \sin nx$	$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$
$(0, l)$	Cosine series	$\frac{a_0}{2} + \sum_1^{\infty} a_n \cos \frac{n\pi x}{l}$	$a_0 = \frac{2}{l} \int_0^l f(x) dx$ $a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx$
$(0, l)$	Sine series	$\sum_1^{\infty} b_n \sin \frac{n\pi x}{l}$	$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$

➤ Complex form of Fourier series of period  $2\pi$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx} \quad \text{where} \quad C_n = \frac{1}{2\pi} \int_c^{c+2\pi} f(x) e^{-inx} dx$$

➤ *Complex form of Fourier series of period 2l*

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/l} \quad \text{where} \quad C_n = \frac{1}{2l} \int_c^{c+2l} f(x) e^{-in\pi x/l} dx$$

➤ *Harmonic Analysis formulae.*

Data :	$x$	$x_1$	$x_2$	$x_3$	...	$x_N$
	$y$	$y_1$	$y_2$	$y_3$	...	$y_N$

**Case-i. (period  $2\pi$ )** Suppose the values of  $x$  are in the interval  $c \leq x < c+2\pi$  or  $c < x \leq c+2\pi$ , Euler's formulae in the modified form are as follows.

$$a_0 = \frac{2}{N} \sum y, \quad a_n = \frac{2}{N} \sum y \cos nx, \quad b_n = \frac{2}{N} \sum y \sin nx$$

**Case-ii. (Period  $2l$ )** Suppose the values of  $x$  are in the interval  $c \leq x < c+2l$  or  $c < x \leq c+2l$ , we have if  $\theta = (\pi x/l)$

$$a_0 = \frac{2}{N} \sum y, \quad a_n = \frac{2}{N} \sum y \cos n\theta, \quad b_n = \frac{2}{N} \sum y \sin n\theta$$

## Unit - II : FOURIER TRANSFORMS

### Definitions at a glance - Infinite Fourier Transforms

Type	Transform	Inverse transform
Fourier transform	$\int_{-\infty}^{\infty} f(x) e^{iux} dx = F(u)$	$\frac{1}{2\pi} \int_{-\infty}^{\infty} F(u) e^{-iux} du = f(x)$
Fourier cosine transform	$\int_0^{\infty} f(x) \cos ux dx = F_c(u)$	$\frac{2}{\pi} \int_0^{\infty} F_c(u) \cos ux du = f(x)$
Fourier sine transform	$\int_{-\infty}^{\infty} f(x) \sin ux dx = F_s(u)$	$\frac{2}{\pi} \int_0^{\infty} F_s(u) \sin ux du = f(x)$

### Unit - III: APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

➤ *Befitting solution for solving standard PDEs with a given set of conditions - Boundary Value Problems [ B.V.P ]*

(i) One dimensional wave equation :  $u_{tt} = c^2 u_{xx}$

$$u(x, t) = (A \cos px + B \sin px) (C \cos cpt + D \sin cpt)$$

(ii) One dimensional heat equation :  $u_t = c^2 u_{xx}$

$$u(x, t) = e^{-c^2 p^2 t} (A \cos px + B \sin px)$$

(iii) Two dimensional Laplace's equation :  $u_{xx} + u_{yy} = 0$

$$u(x, y) = (A \cos px + B \sin px) (C e^{py} + D e^{-py})$$

### Unit - IV : CURVE FITTING & OPTIMIZATION

#### Curve fitting

➤ *Straight line :  $y = ax + b$*

Normal equations are :

$$\sum y = a \sum x + nb \quad [\text{Take } \sum \text{ for the given equation (g.e)}]$$

$$\sum xy = a \sum x^2 + b \sum x \quad [\text{Multiply the g.e by } x \text{ and take } \sum]$$

➤ *Second degree parabola :  $y = ax^2 + bx + c$*

Normal equations are

$$\sum y = a \sum x^2 + b \sum x + nc \quad [\text{Take } \sum \text{ for the g.e}]$$

$$\sum xy = a \sum x^3 + b \sum x^2 + c \sum x \quad [\text{Multiply the g.e by } x \text{ and take } \sum]$$

$$\sum x^2 y = a \sum x^4 + b \sum x^3 + c \sum x^2 \quad [\text{Multiply the g.e by } x^2 \text{ and take } \sum]$$

➤  $y = a e^{bx}$

$$\Rightarrow \log_e y = \log_e a + bx, \text{ since } \log_e e = 1$$

$$\text{ie., } Y = A + BX, \text{ where } Y = \log_e y, A = \log_e a, B = b, X = x.$$

The associated normal equations are

$$\sum Y = nA + B \sum X$$

$$\sum XY = A \sum X + B \sum X^2$$

By solving  $A, B$  are obtained  $A = \log_e a \Rightarrow a = e^A ; b = B$

$$\rightarrow y = a x^b$$

$$\Rightarrow \log_e y = \log_e a + b \log_e x$$

$$\text{ie., } Y = A + BX, \text{ where } Y = \log_e y, A = \log_e a, B = b, X = \log_e x$$

The associated normal equations and the computation of  $a$  and  $b$  are same as the earlier one.

### Optimization

#### ➤ Graphical method

- Constraints are considered in the form of equalities.
- They are put in the intercept form of the straight line  $x/a + y/b = 1$
- These lines along with the coordinate axes forms the boundary of the region known as the convex polygon.
- The value of the objective function is found at all the vertices of the convex polygon to identify the optimum (*maximum or minimum*) value of the objective function.

#### ➤ Simplex method

- The linear inequalities in the constraints are converted into equalities with the introduction of slack/surplus variables.
- Initial simplex tableau is formed along with the indicators which being the coefficients of the variables in the objective function with their sign reversed.
- Simplex method algorithm is carried out and if there are no negative indicators, the process is completed.
- If  $P$  is the objective function for minimization,  $\text{Min. } P = - (\text{Max. value of } -P)$

### Unit - V : NUMERICAL METHODS - 1

➤ *Regula - Falsi formula* for the approximate root of  $f(x) = 0$  lying in  $(a, b)$

$$\text{First approx. } x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

➤ *Newton - Raphson formula* for the approximate root of  $f(x) = 0$  near  $x = x_0$

$$\text{First approx. } x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\text{General formula : } x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

### Unit - Vi : NUMRICAL METHODS - 2

➤ *Finite differences*

$$\Delta f(x) = f(x+h) - f(x) \text{ [Forward difference]}$$

$$\nabla f(x) = f(x) - f(x-h) \text{ [Backward difference]}$$

➤ *Newton's forward interpolation formula*

$$y_r = f(x_0 + rh) = y_0 + r \Delta y_0 + \frac{r(r-1)}{2!} \Delta^2 y_0 + \dots \\ + \frac{r(r-1)(r-2) \cdots (r-n+1)}{n!} \Delta^n y_0$$

$$\text{where } r = (x - x_0)/h$$

➤ *Newton's backward interpolation formula*

$$y_r = f(x_n + rh) = y_n + r \nabla y_n + \frac{r(r+1)}{2!} \nabla^2 y_n + \dots \\ + \frac{r(r+1)(r+2) \cdots (r+n-1)}{n!} \nabla^n y_n$$

$$\text{where } r = (x - x_n)/h$$

➤ *Newton's general (divided difference) interpolation formula*

$$y = f(x) = f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2) \\ + \dots + (x - x_0)(x - x_1) \cdots (x - x_{n-1})f(x_0, x_1, x_2, \dots, x_n)$$

➤ *Lagrange's interpolation formula*

$$y = f(x) = \frac{(x - x_1)(x - x_2) \cdots (x - x_n)y_0}{(x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)} + \frac{(x - x_0)(x - x_2) \cdots (x - x_n)y_1}{(x_1 - x_0)(x_1 - x_2) \cdots (x_1 - x_n)} \\ + \frac{(x - x_0)(x - x_1)(x - x_3) \cdots (x - x_n)y_2}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3) \cdots (x_2 - x_n)} + \dots + \frac{(x - x_0)(x - x_1) \cdots (x - x_{n-1})y_n}{(x_n - x_0)(x_n - x_1) \cdots (x_n - x_{n-1})}$$

➤ Lagrange's inverse interpolation formula

$$x = g(y) = \frac{(y-y_1)(y-y_2)\cdots(y-y_n)x_0}{(y_0-y_1)(y_0-y_2)\cdots(y_0-y_n)} + \frac{(y-y_0)(y-y_2)\cdots(y-y_n)x_1}{(y_1-y_0)(y_1-y_2)\cdots(y_1-y_n)} \\ + \frac{(y-y_0)(y-y_1)(y-y_3)\cdots(y-y_n)x_2}{(y_2-y_0)(y_2-y_1)(y_2-y_3)\cdots(y_2-y_n)} + \cdots + \frac{(y-y_0)(y-y_1)\cdots(y-y_{n-1})x_n}{(y_n-y_0)(y_n-y_1)\cdots(y_n-y_{n-1})}$$

➤ Formulae (Rules) for the evaluation of  $I = \int_a^b y dx$

➤ Simpson's 1/3 rd rule ( $n$  is a multiple of 2)

$$I = \frac{h}{3} \left[ (y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2}) \right]$$

➤ Simpson's 3/8 th rule ( $n$  is a multiple of 3)

$$I = \frac{3h}{8} \left[ (y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}) + 2(y_3 + y_6 + \cdots + y_{n-3}) \right]$$

➤ Weddle's rule ( $n$  is a multiple of 6)

$$I = \frac{3h}{10} \sum_{i=0}^n k y_i$$

where  $k = 1, 5, 1, 6, 1, 5, 2, 5, 1, 6, 1, 5, 2, \dots$

In particular when  $n = 6$  the rule is given by,

$$I = \frac{3h}{10} \left[ y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6 \right]$$

### Unit - VII : NUMERICAL METHODS - 3

➤ Numerical solution of the one dimensional wave equation  $u_{tt} = c^2 u_{xx}$

$$(i) \quad u_{i,1} = \frac{1}{2} \left[ u_{i-1,0} + u_{i+1,0} \right]$$

$$(ii) \quad u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \quad [\text{Explicit formula ; } k = h/c]$$

➤ Numerical solution of the one dimensional heat equation  $u_t = c^2 u_{xx}$

$$(i) \quad u_{i,j+1} = a u_{i-1,j} + (1-2a) u_{i,j} + a u_{i+1,j}$$

[ Schmidt explicit formula valid for  $0 < a \leq 1/2$  ;  $a = kc^2/h^2$

$$(ii) \quad u_{i,j+1} = \frac{1}{2} [u_{i-1,j} + u_{i+1,j}] \text{ when } a = 1/2 \text{ or } k = 2c^2/h^2$$

[ Bendre - Schmidt formula ]

➤ Numerical solution of the Laplace's equation  $u_{xx} + u_{yy} = 0$

$$(i) \quad u_{ij} = \frac{1}{4} [u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1}]$$

[ Standard five point formula ]

$$(ii) \quad u_{i,j} = \frac{1}{4} [u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}]$$

[ Diagonal five point formula ]

### Unit - VIII : DIFFERENCE EQUATIONS & Z - TRANSFORMS

➤ Definition & property

$$Z_T(u_n) = \sum_{n=0}^{\infty} u_n z^{-n} = \bar{u}(z) \quad \& \quad Z_T(k^n u_n) = \bar{u}(z/k)$$

Also  $Z_T(n^k) = -z \frac{d}{dz} Z_T(n^{k-1})$

➤ Z - transform of standard functions

1. $Z_T(1) = \frac{z}{z-1}$	2. $Z_T(k^n) = \frac{z}{z-k}$
3. $Z_T(n) = \frac{z}{(z-1)^2}$	4. $Z_T(k^n n) = k \frac{z}{(z-k)^2}$
5. $Z_T(n^2) = \frac{z^2+z}{(z-1)^3}$	6. $Z_T(k^n n^2) = \frac{kz^2+k^2z}{(z-k)^3}$
7. $Z_T(n^3) = \frac{z^3+4z^2+4z}{(z-1)^4}$	8. $Z_T(k^n n^3) = \frac{kz^3+4k^2z^2+k^3z}{(z-k)^4}$
9. $Z_T[\sin(n\pi/2)] = \frac{z}{z^2+1}$	10. $Z_T[\cos(n\pi/2)] = \frac{z^2}{z^2+1}$



➤ *Initial value theorem*

$$\text{If } Z_T(u_n) = \bar{u}(z) \text{ then } \lim_{z \rightarrow \infty} z \bar{u}(z) = u_0$$

➤ *Final value theorem*

$$\text{If } Z_T(u_n) = \bar{u}(z) \text{ then } \lim_{z \rightarrow 1} [(z-1) \bar{u}(z)] = \lim_{n \rightarrow \infty} u_n$$

### List of standard inverse Z-transforms

$$1. Z_T^{-1} \left[ \frac{z}{z-1} \right] = 1$$

$$2. Z_T^{-1} \left[ \frac{z}{z-k} \right] = k^n$$

$$3. Z_T^{-1} \left[ \frac{z}{(z-1)^2} \right] = n$$

$$4. Z_T^{-1} \left[ k \frac{z}{(z-k)^2} \right] = k^n \cdot n^n$$

$$5. Z_T^{-1} \left[ \frac{z^2+z}{(z-1)^3} \right] = n^2$$

$$6. Z_T^{-1} \left[ \frac{kz^2+k^2z}{(z-k)^3} \right] = k^n \cdot n^2$$

$$7. Z_T^{-1} \left[ \frac{z^3+4z^2+z}{(z-1)^4} \right] = n^3$$

$$8. Z_T^{-1} \left[ \frac{kz^3+4k^2z^2+k^3z}{(z-k)^4} \right] = k^n \cdot n^3$$

$$9. Z_T^{-1} \left[ \frac{z}{z^2+1} \right] = \sin(n\pi/2)$$

$$10. Z_T^{-1} \left[ \frac{z^2}{z^2+1} \right] = \cos(n\pi/2)$$

➤ *Some useful expressions for solving difference equations using Z-transforms.*

$$(i) Z_T(u_{n+1}) = z [\bar{u}(z) - u_0]$$

$$(ii) Z_T(u_{n+2}) = z^2 [\bar{u}(z) - u_0 - u_1 z^{-1}]$$

$$(iii) Z_T(u_{n+3}) = z^3 [\bar{u}(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}]$$

**Names of the Greek letters (Alphabets)  
used in science along with their commonly used  
capital letters (given in the bracket).**

	Name	Letter/Alphabet		Name	Letter/Alphabet
1.	Alpha	$\alpha$	12.	Mu	$\mu$
2.	Beta	$\beta$	13.	Nu	$\nu$
3.	Gamma	$\gamma$ ( $\Gamma$ )	14.	Tau	$\tau$
4.	Delta	$\delta$ ( $\Delta$ )	15.	Rho	$\rho$
5.	Lambda	$\lambda$ ( $\Lambda$ )	16.	Phi	$\phi$
6.	Sigma	$\sigma$ ( $\Sigma$ )	17.	Psi	$\psi$
7.	Omega	$\omega$ ( $\Omega$ )	18.	Chi	$\chi$
8.	Kappa	$\kappa$	19.	Pi	$\pi$
9.	Theta	$\theta$ ( $\Theta$ )	20.	Xi	$\xi$
10.	Eta	$\eta$	21.	Epsilon	$\epsilon$
11.	Zeta	$\zeta$			

Some Notations

- $\in$  ... Belongs to
- $\notin$  ... Doesnot belong to
- $\exists$  ... There exists
- $\forall$  ... For all
- $\Rightarrow$  ... Implies
- $\Leftrightarrow$  ... Implies & implied by
- $\ni$  / ... Such that
- $\cup$  ... Union
- $\cap$  ... Intersection

## ALPHABETICAL INDEX

<b>A</b>	
Algebraic equation . . . . .	251
<b>B</b>	
Backward differences . . . . .	297, 299
Basic/Basic feasible solution . . . . .	234
Basic & non basic variables . . . . .	234
Bendre - Schmidt formula . . . . .	388
<b>C</b>	
Complex Fourier series . . . . .	91
Complex Fourier transform . . . . .	119
Constraints . . . . .	215
Convex polygon . . . . .	216
Curve fitting . . . . .	195
exponential curves . . . . .	197 & 198
parabola . . . . .	196
straight line . . . . .	195
<b>D</b>	
D'Alembert's solution . . . . .	189
Damping rule . . . . .	419
Decision variables . . . . .	215
Departing & entering variable . . . . .	237
Diagonally dominant system . . . . .	274
Diagonal five point formula . . . . .	399
Difference equation . . . . .	415, 455
Dirichlet's conditions . . . . .	5
Divided differences . . . . .	325
<b>E</b>	
Eigen values & eigen vectors . . . . .	289
Elliptic p.d.e . . . . .	367
Euler's formulae . . . . .	3
Even function . . . . .	32, 45
Explicit formula . . . . .	371
Extrapolation . . . . .	305
<b>F</b>	
False position method . . . . .	252
Feasible region/solution . . . . .	216
Final value theorem . . . . .	438
Finite differences . . . . .	297, 299
Forward differences . . . . .	297, 299
Fourier series . . . . .	1, 5
Fourier transforms . . . . .	119, 122
(complex, cosine, sine)	
<b>G</b>	
Gauss-Seidel method . . . . .	274
Graphical method . . . . .	216
<b>H</b>	
Half range Fourier series . . . . .	73
Harmonic analysis . . . . .	96
Heat equation . . . . .	160, 367, 387
Hyperbolic p.d.e . . . . .	367
<b>I</b>	
Indicators . . . . .	236
Initial value theorem . . . . .	437
Interpolation . . . . .	305
Inverse Fourier transform . . . . .	119, 122
Inverse Z-transform . . . . .	445
<b>L</b>	
Lagrange's interpolation formula . . . . .	336
Laplace's equation . . . . .	161, 367, 398
Least squares method . . . . .	195
Liebman's iteration process . . . . .	400
Linear programming . . . . .	215
<b>M</b>	
Modulation properties . . . . .	121, 122
<b>N</b>	
Newton's interpolation formulae	
forward & backward . . . . .	306
general/divided difference . . . . .	326
Newton-Raphson method . . . . .	261
Numerical differentiation . . . . .	346
Numerical integration . . . . .	350

Numerical solution of algebraic & transcendental equations . . . .	251
Numerical solution of p.d.e . . . .	367
heat equation . . . . .	387
Laplace's equation . . . . .	398
wave equation . . . . .	370

**O**

Objective function . . . . .	215
Odd function . . . . .	32, 45
Optimal solution . . . . .	216
Optimization . . . . .	215

**P**

Partial differential equations . . . . .	157, 367
Pivotal column . . . . .	236

**R**

Rayleigh's power method . . . . .	289
Regula falsi method . . . . .	252

**S**

Schmidt's explicit formula . . . . .	387
Separation of variables method . . . . .	157
Slack variables . . . . .	233
Simplex algorithm/method . . . . .	233
Simpson's rule . . . . .	350
(one third & three eighth)	
Standard five point formula . . . . .	399
Standard form of LPP . . . . .	233
Surplus variables . . . . .	233

**U**

Unbounded solution . . . . .	216
------------------------------	-----

**W**

Wave equation . . . . .	158, 367, 370
Weddle's rule . . . . .	351

**Z**

Z-transform . . . . .	415, 416
-----------------------	----------

