BEATING THE MEMORY

[Formulae, Properties and Results to be remembered from all the chapters at a glance]

Unit - I : FOURIER SERIES

- *Fourier series of period* 2π *and Euler's formulae for the Fourier coefficients* a_0, a_n, b_n

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \ldots \qquad \text{[Fourier series]}$$

$$a_0 = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x) \sin nx\, dx \qquad \text{[Euler's formulae]}$$

- *Fourier series of arbitrary period* $2l$ *and the related Euler's formulae*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \qquad \text{[Fourier series]}$$

$$a_0 = \frac{1}{l} \int_{c}^{c+2l} f(x)\, dx$$

$$a_n = \frac{1}{l} \int_{c}^{c+2l} f(x) \cos \frac{n\pi x}{l}\, dx$$

$$b_n = \frac{1}{l} \int_{c}^{c+2l} f(x) \sin \frac{n\pi x}{l}\, dx \ldots \qquad \text{[Euler's formulae]}$$

- *Fourier coefficients in the case of even and odd nature of* $f(x)$.

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Interval	$f(x)$ is even			$f(x)$ is odd		
	a_0	a_n	b_n	a_0	a_n	b_n
$(-\pi, \pi)$ or $(0, 2\pi)$	$\frac{2}{\pi}\int_0^{\pi} f(x)\,dx$	$\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$	0	0	0	$\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$
$(-l, l)$ or $(0, 2l)$ type of intervals	$\frac{2}{l}\int_0^{l} f(x)\,dx$	$\frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx$	0	0	0	$\frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}\,dx$

Note : $f(x)$ is even implies $b_n = 0$ and $f(x)$ is odd implies $a_0 = 0 = a_n$

➢ *Half range Fourier series (cosine / sine) along with the related formulae for the Fourier coefficients.*

$f(x)$ in	Requirement	Series	Fourier Coefficients
$(0, \pi)$	Cosine series	$\frac{a_0}{2} + \sum_1^{\infty} a_n \cos nx$	$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$ $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$
$(0, \pi)$	Sine series	$\sum_1^{\infty} b_n \sin nx$	$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$
$(0, l)$	Cosine series	$\frac{a_0}{2} + \sum_1^{\infty} a_n \cos\frac{n\pi x}{l}$	$a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx$ $a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx$
$(0, l)$	Sine series	$\sum_1^{\infty} b_n \sin\frac{n\pi x}{l}$	$b_n = \frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}\,dx$

➢ *Complex form of Fourier series of period* 2π

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx} \qquad \text{where} \quad C_n = \frac{1}{2\pi}\int_c^{c+2\pi} f(x)\, e^{-inx}\,dx$$

➢ *Complex form of Fourier series of period* $2l$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n \, e^{i n \pi x/l} \quad \text{where} \quad C_n = \frac{1}{2l} \int_c^{c+2l} f(x)\, e^{-i n \pi x/l} \, dx$$

➢ *Harmonic Analysis formulae.*

Data :

x	x_1	x_2	x_3	...	x_N
y	y_1	y_2	y_3	...	y_N

Case-i. (period 2π) Suppose the values of x are in the interval $c \le x < c+2\pi$ or $c < x \le c+2\pi$, Euler's formulae in the modified form are as follows.

$$a_0 = \frac{2}{N} \sum y, \quad a_n = \frac{2}{N} \sum y \cos nx, \quad b_n = \frac{2}{N} \sum y \sin nx$$

Case-ii. (Period $2l$) Suppose the values of x are in the interval $c \le x < c+2l$ or $c < x \le c+2l$, we have if $\theta = (\pi x/l)$

$$a_0 = \frac{2}{N} \sum y, \quad a_n = \frac{2}{N} \sum y \cos n\theta, \quad b_n = \frac{2}{N} \sum y \sin n\theta$$

Unit - II : FOURIER TRANSFORMS

Definitions at a glance - Infinite Fourier Transforms

Type	Transform	Inverse transform
Fourier transform	$\int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx = F(u)$	$\frac{1}{2\pi} \int_{-\infty}^{\infty} F(u)\, e^{-iux}\, du = f(x)$
Fourier cosine transform	$\int_0^{\infty} f(x) \cos ux \, dx = F_c(u)$	$\frac{2}{\pi} \int_0^{\infty} F_c(u) \cos ux \, du = f(x)$
Fourier sine transform	$\int_{-\infty}^{\infty} f(x) \sin ux \, dx = F_s(u)$	$\frac{2}{\pi} \int_0^{\infty} F_s(u) \sin ux \, du = f(x)$

Unit - III : APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

➢ *Befitting solution for solving standard PDEs with a given set of conditions - Boundary Value Problems [B.V.P]*

(i) One dimensional wave equation : $u_{tt} = c^2 u_{xx}$

$$u(x, t) = (A \cos px + B \sin px)\ (C \cos cpt + D \sin cpt)$$

(ii) One dimensional heat equation : $u_t = c^2 u_{xx}$

$$u(x, t) = e^{-c^2 p^2 t}(A \cos px + B \sin px)$$

(iii) Two dimensional Laplace's equation : $u_{xx} + u_{yy} = 0$

$$u(x, y) = (A \cos px + B \sin px)\ (C e^{py} + D e^{-py})$$

Unit - IV : CURVE FITTING & OPTIMIZATION

Curve fitting

➢ *Straight line* : $y = ax + b$

Normal equations are :

$\sum y = a \sum x + nb$ [Take '$\sum$' for the given equation (g.e)]

$\sum xy = a \sum x^2 + b \sum x$ [Multiply the g.e by x and take '$\sum$']

➢ *Second degree parabola* : $y = ax^2 + bx + c$

Normal equations are

$\sum y = a \sum x^2 + b \sum x + nc$ [Take '$\sum$' for the g.e]

$\sum xy = a \sum x^3 + b \sum x^2 + c \sum x$ [Multiply the g.e by x and take '$\sum$']

$\sum x^2 y = a \sum x^4 + b \sum x^3 + c \sum x^3 + c \sum x^2$ [Multiply the g.e by x^2 and take '$\sum$']

➢ $y = a e^{bx}$

$\Rightarrow$ $\log_e y = \log_e a + bx$, since $\log_e e = 1$

ie., $Y = A + BX$, where $Y = \log_e y$, $A = \log_e a$, $B = b$, $X = x$.

The associated normal equations are

$\sum Y = nA + B \sum X$

$\sum XY = A \sum X + B \sum X^2$

By solving A, B are obtained $A = \log_e a \Rightarrow a = e^A$; $b = B$

➢ $y = a x^b$

$\Rightarrow$ $\log_e y = \log_e a + b \log_e x$

ie., $Y = A + BX$, where $Y = \log_e y$, $A = \log_e a$, $B = b$, $X = \log_e x$

The associated normal equations and the computation of a and b are same as the earlier one.

Optimization

➢ *Graphical method*

- Constraints are considered in the form of equalities.
- They are put in the intercept form of the straight line $x/a + y/b = 1$
- These lines along with the coordinate axes forms the boundary of the region known as the convex polygon.
- The value of the objective function is found at all the vertices of the convex polygon to identify the optimum (*maximum or minimum*) value of the objective function.

➢ *Simplex method*

- The linear inequalities in the constraints are converted into equalities with the introduction of slack/surplus variables.
- Initial simplex tableau is formed along with the indicators which being the coefficients of the variables in the objective function with their sign reversed.
- Simplex method algorithm is carried out and if there are no negative indicators, the process is completed.
- If P is the objective function for minimization, Min. $P = -$ (Max. value of $-P$)

Unit - V : NUMERICAL METHODS - 1

➢ *Regula - Falsi formula* for the approximate root of $f(x) = 0$ lying in (a, b)

First approx. $x_1 = \dfrac{a f(b) - b f(a)}{f(b) - f(a)}$

➢ *Newton - Raphson formula* for the approximate root of $f(x) = 0$ near $x = x_0$

First approx. $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

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General formula : $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Unit - VI : NUMRICAL METHODS - 2

- *Finite differences*

$\Delta f(x) = f(x+h) - f(x)$ [Forward difference]

$\nabla f(x) = f(x) - f(x-h)$ [Backward difference]

- *Newton's forward interpolation formula*

$$y_r = f(x_0 + rh) = y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \cdots + \frac{r(r-1)(r-2)\cdots(r-\overline{n-1})}{n!}\Delta^n y_0$$

where $r = (x - x_0)/h$

- *Newton's backward interpolation formula*

$$y_r = f(x_n + rh) = y_n + r\,\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \cdots + \frac{r(r+1)(r+2)\cdots(r+\overline{n-1})}{n!}\nabla^n y_n$$

where $r = (x - x_n)/h$

- *Newton's general (divided difference) interpolation formula*

$$y = f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2) + \cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1}) f(x_0, x_1, x_2, \cdots x_n)$$

- *Lagrange's interpolation formula*

$$y = f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)\,y_0}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)\,y_1}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} + \frac{(x-x_0)(x-x_1)(x-x_3)\cdots(x-x_n)\,y_2}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n)} + \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})\,y_n}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}$$

➢ *Lagrange's inverse interpolation formula*

$$x = g(y) = \frac{(y-y_1)(y-y_2)\cdots(y-y_n)\,x_0}{(y_0-y_1)(y_0-y_2)\cdots(y_0-y_n)} + \frac{(y-y_0)(y-y_2)\cdots(y-y_n)\,x_1}{(y_1-y_0)(y_1-y_2)\cdots(y_1-y_n)}$$
$$+ \frac{(y-y_0)(y-y_1)(y-y_3)\cdots(y-y_n)\,x_2}{(y_2-y_0)(y_2-y_1)(y_2-y_3)\cdots(y_2-y_n)} + \cdots + \frac{(y-y_0)(y-y_1)\cdots(y-y_{n-1})\,x_n}{(y_n-y_0)(y_n-y_1)\cdots(y_n-y_{n-1})}$$

➢ **Formulae (Rules) for the evaluation of** $I = \int_a^b y\,dx$

➢ *Simpson's 1/3 rd rule* (n is a multiple of 2)

$$I = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+\cdots+y_{n-1})+2(y_2+y_4+\cdots+y_{n-2})\right]$$

➢ *Simpson's 3/8 th rule* (n is a multiple of 3)

$$I = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\cdots+y_{n-1})+2(y_3+y_6+\cdots+y_{n-3})\right]$$

➢ *Weddle's rule* (n is a multiple of 6)

$$I = \frac{3h}{10}\sum_{i=0}^{n} k\,y_i$$

where $k = 1, 5, 1, 6, 1, 5, 2, 5, 1, 6, 1, 5, 2, \cdots$

In particular when $n = 6$ the rule is given by,

$$I = \frac{3h}{10}\left[y_0+5y_1+y_2+6y_3+y_4+5y_5+y_6\right]$$

Unit - VII : NUMERICAL METHODS - 3

➢ *Numerical solution of the one dimensional wave equation* $u_{tt} = c^2\,u_{xx}$

(i) $$u_{i,1} = \frac{1}{2}\left[u_{i-1,0} + u_{i+1,0}\right]$$

(ii) $$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$ [Explicit formula ; $k = h/c$]

➢ *Numerical solution of the one dimensional heat equation* $u_t = c^2\,u_{xx}$

(i) $$u_{i,j+1} = a\,u_{i-1,j} + (1-2a)\,u_{i,j} + a\,u_{i+1,j}$$

[Schmidt explicit formula valid for $0 < a \le 1/2$; $a = kc^2/h^2$

(ii) $u_{i,\,j+1} = \frac{1}{2}\left[u_{i-1,\,j} + u_{i+1,\,j} \right]$ when $a = 1/2$ or $k = 2c^2/h^2$

[Bendre - Schmidt formula]

➢ *Numerical solution of the Laplace's equation* $u_{xx} + u_{yy} = 0$

(i) $u_{ij} = \frac{1}{4}\left[u_{i-1,\,j} + u_{i+1,\,j} + u_{i,\,j+1} + u_{i,\,j-1} \right]$

[Standard five point formula]

(ii) $u_{i,\,j} = \frac{1}{4}\left[u_{i-1,\,j+1} + u_{i+1,\,j-1} + u_{i+1,\,j+1} + u_{i-1,\,j-1} \right]$

[Diagonal five point formula]

Unit - VIII : DIFFERENCE EQUATIONS & Z - TRANSFORMS

➢ *Definition & property*

$$Z_T(u_n) = \sum_{n=0}^{\infty} u_n z^{-n} = \bar{u}(z) \quad \& \quad Z_T(k^n u_n) = \bar{u}(z/k)$$

Also $Z_T(n^k) = -z \frac{d}{dz} Z_T(n^{k-1})$

➢ *Z - transform of standard functions*

1. $Z_T(1) = \frac{z}{z-1}$	2. $Z_T(k^n) = \frac{z}{z-k}$
3. $Z_T(n) = \frac{z}{(z-1)^2}$	4. $Z_T(k^n n) = k\frac{z}{(z-k)^2}$
5. $Z_T(n^2) = \frac{z^2+z}{(z-1)^3}$	6. $Z_T(k^n n^2) = \frac{kz^2+k^2z}{(z-k)^3}$
7. $Z_T(n^3) = \frac{z^3+4z^2+4z}{(z-1)^4}$	8. $Z_T(k^n n^3) = \frac{kz^3+4k^2z^2+k^3z}{(z-k)^4}$
9. $Z_T[\sin(n\pi/2)] = \frac{z}{z^2+1}$	10. $Z_T[\cos(n\pi/2)] = \frac{z^2}{z^2+1}$

➤ *Initial value theorem*

If $Z_T(u_n) = \bar{u}(z)$ then $\lim_{z \to \infty} \bar{u}(z) = u_0$

➤ *Final value theorem*

If $Z_T(u_n) = \bar{u}(z)$ then $\lim_{z \to 1} [(z-1)\,\bar{u}(z)] = \lim_{n \to \infty} u_n$

List of standard inverse Z-transforms

1. $Z_T^{-1}\left[\dfrac{z}{z-1}\right] = 1$

2. $Z_T^{-1}\left[\dfrac{z}{z-k}\right] = k^n$

3. $Z_T^{-1}\left[\dfrac{z}{(z-1)^2}\right] = n$

4. $Z_T^{-1}\left[k\dfrac{z}{(z-k)^2}\right] = k^n \cdot n^n$

5. $Z_T^{-1}\left[\dfrac{z^2+z}{(z-1)^3}\right] = n^2$

6. $Z_T^{-1}\left[\dfrac{kz^2+k^2z}{(z-k)^3}\right] = k^n \cdot n^2$

7. $Z_T^{-1}\left[\dfrac{z^3+4z^2+z}{(z-1)^4}\right] = n^3$

8. $Z_T^{-1}\left[\dfrac{kz^3+4k^2z^2+k^3z}{(z-k)^4}\right] = k^n \cdot n^3$

9. $Z_T^{-1}\left[\dfrac{z}{z^2+1}\right] = \sin(n\pi/2)$

10. $Z_T^{-1}\left[\dfrac{z^2}{z^2+1}\right] = \cos(n\pi/2)$

➤ *Some useful expressions for solving difference equations using Z-transforms.*

(i) $Z_T(u_{n+1}) = z\left[\bar{u}(z) - u_0\right]$

(ii) $Z_T(u_{n+2}) = z^2\left[\bar{u}(z) - u_0 - u_1 z^{-1}\right]$

(iii) $Z_T(u_{n+3}) = z^3\left[\bar{u}(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}\right]$

Names of the Greek letters (Alphabets) used in science along with their commonly used capital letters (given in the bracket).

	Name	Letter/Alphabet		Name	Letter/Alphabet
1.	Alpha	α	12.	Mu	μ
2.	Beta	β	13.	Nu	ν
3.	Gamma	γ (Γ)	14.	Tau	τ
4.	Delta	δ (Δ)	15.	Rho	ρ
5.	Lambda	λ (Λ)	16.	Phi	φ
6.	Sigma	σ (Σ)	17.	Psi	ψ
7.	Omega	ω (Ω)	18.	Chi	χ
8.	Kappa	κ	19.	Pi	π
9.	Theta	θ (Θ)	20.	Xi	ξ
10.	Eta	η	21.	Epsilon	ε
11.	Zeta	ζ			

Some Notations

∈ ... Belongs to

∉ ... Doesnot belong to

∃ ... There exists

∀ ... For all

⇒ ... Implies

⇔ ... Implies & implied by

∋ / ... Such that

∪ ... Union

∩ ... Intersection

ALPHABETICAL INDEX